高愛迪斯 第五十三期【數學題目】

1120302

100 +1

夾心數字

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聽說高愛迪斯最喜歡吃的零食就是夾心餅乾了，尤其是仁愛食品公司製作的夾心餅乾更是愛不釋手。在一個陽光燦爛的早上，高愛迪斯突然靈光一閃，就把手上的 6 塊餅乾編成 1-3 號各兩塊，並且把它們重新打亂再排成一排，結果居然發現 2 個編號 1 的餅乾中間夾了 1 塊餅乾，2 個編號 2 的餅乾中間夾了 2 塊餅乾，2 個編號 3 的餅乾中間夾了 3 塊餅乾(如下圖一所示)，真是太神奇了。高愛迪斯還特別將它們稱為【夾心數字】。可是高愛迪斯並不確定 1-10 所有的數都能排出夾心數字，像 2 個 1，2 個 2 就無法成功，所以當你無法成功時請詳細說明無法成功的原因。

【圖一】

3 1 2 1 3 2

【問題一】請使用數字1-4各2個，共8個數字，排入下列方格中，並且符合2個1中間夾1個數字，2個2中間夾2個數字，2個3中間夾3個數字，2個4中間夾4個數字(1★)

4	1	3	1	2	4	3	2

【問題二】請使用數字1-5各2個，共10個數字，排入下列方格中，並且符合2個1中間夾1個數字，2個2中間夾2個數字，2個3中間夾3個數字，2個4中間夾4個數字，以此類推……(1★)

原因： 無法解(請見後面報告)

【問題三】請使用數字1-6各2個，共12個數字，排入下列方格中，並且符合2個1中間夾1個數字，2個2中間夾2個數字，2個3中間夾3個數字，2個4中間夾4個數字，以此類推……(1★)

原因：無法解〔請見後面報告〕

【問題四】請使用數字1-7各2個，共14個數字，排入下列方格中，並且符合2個1中間夾1個數字，2個2中間夾2個數字，2個3中間夾3個數字，2個4中間夾4個數字(1★)

4	5	6	7	1	4	1	5	3	6	2	7	3	2

+1

原因：〔請見後面報告〕

+1

【問題五】請使用數字1-8各2個，共16個數字，排入下列方格中，並且符合2個1中間夾1個數字，2個2中間夾2個數字，2個3中間夾3個數字，2個4中間夾4個數字，以此類推……(1★)

4	5	6	7	8	4	1	5	1	6	3	7	2	8	3	2

原因：〔請見後面報告〕

【問題六】請使用數字1-9各2個，共18個數字，排入下列方格中，並且符合2個1中間夾1個數字，2個2中間夾2個數字，2個3中間夾3個數字，2個4中間夾4個數字，以此類推……(1★)

原因：無法解[請見後面報告]

【問題七】請使用數字1-10各2個，共20個數字，排入下列方格中，並且符合2個1中間夾1個數字，2個2中間夾2個數字，2個3中間夾3個數字，2個4中間夾4個數字，以此類推……(1★)

原因：無法解[請見後面報告]

【問題八】研究結果與發現:(數字排列的技巧、特殊的限制、有無規律…)1★

(請見後面報告)

【問題九】你能排出 10 以上的夾心數字嗎?任意排出 1 個不同的數即獲得 1 顆星(最多獲得 2★)

11 => 1,2,1,11,2,5,7,10,8,3,9,5,6,3,7,11,4,8,10,6,9,4

12 => 12,10,8,6,4,11,9,7,5,4,6,8,10,12,5,7,9,11,3,1,2,1,3,2

截止日期：112 年 4 月 28 日(星期五)下午 4：00

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高愛迪斯第 53 期

介紹夾心數字起原

這個高愛迪斯作業是在介紹『Langford 數字』，它是被一個叫 C. Dudley Langford 的數學家在看他兒子玩積木時意外發現的。他的兒子把積木排成：兩個黃色積木之間有了三個積木，兩個藍色積木之間有了兩個積木，兩個紅色積木之間有了一個積木。Langford 發現了這個有趣的規律，於是就在 1958 年的一個數學雜誌上發表了他的發現。

後來，其他的數學家也開始研究這個規律，想要尋找為甚麼有些數字可以排出規律，有些數字不行。那些數學家們也嘗試了更大的數字來解開這個題目。我會用一些我自己的和那些數學家的觀察來解釋為甚麼有些數字排不出夾心數字，我也會討論可以排出來的夾心數字有哪些規律。

解釋夾心數字規律、公式

如何知道那些數字有夾心數字，那些沒有呢?

$n=$ 題目中最大的數

例如：2 3 4 2 1 3 1 4　$n=4$

一排有 $2n$ 個格子

格子編號：$1,2,3,\ldots,2n$

所有格子編號的合:

$=1+2+3+\cdots+2n$

請看附錄 1 算合的方式 $x=2n$

$$1+2+3+\cdots+2n=\frac{(2n)^2+2n}{2}$$

簡化算式：

$$1+2+3+\cdots+2n=\frac{4n^2+2n}{2}=\boxed{2n^2+n}$$

格子編號與格子中的數字的代號：

夾心數字	2	3	4	2	1	3	1	4
格子編號	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*

$r =$ 格子內的數字

$A_r =$ 數字第一次出現

$B_r =$ 數字第二次出現

$A_2 =$ 數字 2 出現第一次的格子$\left(\text{由左至右}\right) = 1$

$B_2 =$ 數字 2 出現第二次的格子$\left(\text{由左至右}\right) = 4$

$A_3 =$ 數字 3 出現第一次的格子$\left(\text{由左至右}\right) = 2$

$B_3 =$ 數字 3 出現第二次的格子$\left(\text{由左至右}\right) = 6$

我們可以發現一個規律(請猜考以下表格)：

r 格子內的數字	A_r 在的格子編號	B_r 在的格子編號	A_r 和 B_r 之間的距離
1	5	7	2
2	1	4	3
3	2	6	4
4	3	8	5

$A_r \leftarrow (r+1) \rightarrow B_r \qquad B_r = A_r + r + 1$

現在我們把所有的格子編號都加起來

$$
\begin{aligned}
&= (A_1 + B_1) + (A_2 + B_2) + (A_3 + B_3) + \cdots + (A_n + B_n) \\
&= [A_1 + (A_1 + 1 + 1)] + [A_2 + (A_2 + 2 + 1)] + [A_3 + (A_3 + 3 + 1)] + \cdots \\
&\qquad + [A_n + (A_n + n + 1)] \\
&= (2 \times A_1 + 2 \times A_2 + \cdots + 2 \times A_n) + (1 + 2 + \cdots + n) \\
&\quad + (1 + 1 + \cdots + 1) \\
&= 2 \times (A_1 + A_2 + \cdots + A_n) + \frac{n^2 + n}{2} + n
\end{aligned}
$$

如何從 1 加到n，如何解出黃色螢光的地方〔詳細請見附錄 1〕

我們簡化一下公式，把$A_1 + A_2 + \cdots + A_n$ 變成 S

$S = A_1 + A_2 + \cdots + A_n$

$$2 \times (A_1 + A_2 + \cdots + A_n) + \frac{n^2+n}{2} + n = \boxed{2S + \frac{n^2+n}{2} + n}$$

現在我們有兩種方式可以把所有的格子編號加起來

第一種方式 $= \boxed{2n^2 + n}$　　第二種方式 $= \boxed{2S + \frac{n^2+n}{2} + n}$

我們現在讓這兩種加起來所有格子編號的公式相等

$$\boxed{2n^2 + n} = \boxed{2S + \frac{n^2+n}{2} + n}$$

把兩種公式都× 2

$$4n^2 + 2n = 4S + n^2 + n + 2n$$

兩邊都加 n^2

$$4n^2 - n^2 + 2n = 4S + n^2 - n^2 + n + 2n$$
$$3n^2 + 2n = 4S + n + 2n$$

兩邊都減 $3n$

$$3n^2 + 2n - 3n = 4S + n + 2n - 3n$$
$$3n^2 - n = 4S$$

記得 S 是一個整數，所以：

若 $3n^2 - n = 4$ 的倍數，題目就有可能成功排出正確的順序

幫數字分類：

因為沒有辦法每個數字都試，所以要找出一種幫數字分類的分法，看看那些數字有夾心數字，那些數字沒有夾心數字。

我在寫完一些題目之後發現如果最大的數字是 5、6、9、10、13、14，就不能排出夾心數字。而且我發現這些數字都等於四的倍數加一或加二，所以我猜測假如最大的數字是四的倍數加一或加二，就排不出夾心數字了。以下我要解釋：

為甚麼最大的數 = 4 的倍數 + 1 或 + 2 時，沒有辦法排出成功的順序

n 的公式(k =任何一整數)	**Ex**	**除以 4 以後可能會出現的餘數**
$4k$	$4, 8, 12$	0
$4k + 1$	$5, 9, 13$	1
$4k + 2$	$6, 10, 14$	2
$4k + 3$	$7, 11, 15$	3

n 的公式這一欄的數字($4k$、$4k + 1$、$4k + 2$、$4k + 3$) 已經包括了所有的數字(1-無限)，因為如果繼續加上去就是$4k + 4$也等於 4k。

若... $n = 4k$

$$3n^2 - n = 3(4k)^2 - 4k = 48k^2 - 4k$$

把兩個公式都 ÷ 4 來看看是不是符合 $3n^2 - n$ 是四的倍數

$$\frac{48k^2 - 4k}{4} = 12k^2 - k$$

因為$48k^2$ 和$4k$都是 4 的倍數，所以若$n = 4k$，就有可能排出成功的順序

若... $n = 4k + 1$

$$3n^2 - n = 3(4k+1)^2 - (4k+1)$$
$$= 3(4k+1)(4k+1) - (4k+1)$$

如何簡化 $(a+b) \times (c+d)$ ，請參考附錄 2

$$3n^2 - n = 3(16k^2 + 4k + 4k + 1) - (4k+1)$$
$$= 48k^2 + 24k + 3 - 4k - 1$$
$$= 48k^2 + 20k + 2$$

把兩個公式都 ÷ 4

$$\frac{48k^2 + 20k + 2}{4} = 12k^2 + 5k + \frac{1}{2}$$

雖然$48k^2$和 $20k$ 都是 4 的倍數,但是 2 不是 4 的倍數，所以若$n = 4k + 1$，那麼就無法排出成功的順序

若... $n = 4k + 2$

$$3n^2 - n = 3(4k+2)^2 - (4k+2)$$
$$= 3(4k+2)(4k+2) - (4k+2)$$
$$3n^2 - n = 3(16k^2 + 8k + 8k + 4) - (4k+2)$$
$$= 48k^2 + 48k + 12 - 4k - 2$$
$$= 48k^2 + 44k + 10$$

把兩個公式都 ÷ 4

$$\frac{48k^2 + 44k + 10}{4} = 12k^2 + 11k + \frac{5}{2}$$

雖然$48k^2$和 $44k$ 都是 4 的倍數,但是 10 不是 4 的倍數，所以若$n = 4k + 2$，那麼就無法排出成功的順序

若… $n = 4k + 3$

$$3n^2 - n = 3(4k+3)^2 - (4k+3)$$
$$= 3(4k+3)(4k+3) - (4k+3)$$
$$3n^2 - n = 3(16k^2 + 12k + 12k + 9) - (4k+3)$$
$$3n^2 - n = 48k^2 + 72k + 27 - 4k - 3$$
$$= 48k^2 + 68k + 24$$

把兩個公式都 ÷ 4

$$\frac{48k^2 + 68k + 24}{4} = 12k^2 + 17k + 6$$

因為$48k^2$、$68k$和 24 都是 4 的倍數，所以若$n = 4k$，就有可能排出成功的順序

總結：

若 $n = 4k$ or 或$n = 4k + 3$ 就有可能排出成功的順序

若$n = 4k + 1$ or 或$n = 4k + 2$ 就不可能排出成功的順序

可以排出夾心數字題目的規律、排法

在可以成功的題目中我發現了一個規律：

若 n 除 2 是整數，那麼我們稱商為 y

若 n 除 2 不是整數，那麼我們稱【商 +0.5 】為 y

我發現的規律如下:

y	$y+1$	$y+2$	$y+3$	…	…	n			

第一個格子必須是 y，再來是$y+1, y+2,,,,,, n$

再來就按照規則的間隔把第二個 y 寫上去，第二個 $y+1$ 寫上去，第二個 $y+2$ 寫上去……最後把第二個 n 寫上去

因為 y 到 n 都寫了，所以剩下來的格子就是要寫$< y$的數字。(可以看到靠左邊的空格會呈現一個有數字，一個沒數字的狀態，所以靠左邊的空格就無法放入雙數〔詳細請見下圖〕)。

4 (y)	5 ($y+1$)	6 ($y+2$)	7 (n)		4		5		6		7		
	↑ 2 ✗			↑ 2			↑ 2 ✗						

如果把 2(雙數)放在左邊格子裡，2 可以放的 2 個位置都已經有數字了。所以這些格子只能放單數。

接下來我要證明為甚麼雙數可以放在右邊的格子〔請見下圖〕

4 (y)	5 ($y+1$)	6 ($y+2$)	7 (n)		4		5		6		7		
										↑ 2 ✓			↑ 2

如果把 2(雙數)放在右邊的格子裡，2 可以放的位置上還沒有數字。所以在填空格時，最左邊的格子要填單數，最右邊的格子要優先填入雙數。

最後：數字越大，可以排出的夾心數字順序就越多，所以不是所有夾心數字都只有一種排法。

附錄 1

如何從 1 加到 100（1 加到 x）

先算【1 加到 100】加【100 加到 1】

	1	2	3	…	99	100
+	100	99	98	…	2	1
	101	101	101	…	101	101

上面的直式也就代表:

$$101 \times 100 = 10100$$

$$10100 \div 2 = 5050$$ ← ÷ 2 才等於 1 加到 100

$$1 + 2 + 3 + \cdots + 99 + 100 = 5050$$

$$1 + 2 + 3 + \cdots + 99 + 100 = 100 \times 101 \div 2$$

$$1 + 2 + 3 + \cdots + (x - 1) + x = x \times (x + 1) \div 2$$

$$1 + 2 + 3 + \cdots + (x - 1) + x = \frac{x(x + 1)}{2}$$

$$= \frac{x^2 + x}{2}$$

附錄 2

如何簡化$(a + b) \times (c + d)$:

$$(a + b) \times (c + d) = a \times c + a \times d + b \times c + b \times d$$

$$(a + b)(c + d) = ac + ad + bc + bd$$

非常棒 解說清楚 資料詳盡

5/10

參考資料

1. Problem. (1958). *The Mathematical Gazette*, *42*(341), 228–228. https://doi.org/10.2307/3610395
2. C. J. Priday, & Roy O. Davies. (1959). On Langford's Problem. *The Mathematical Gazette*, *43*(346), 250–255. https://doi.org/10.2307/3610650
3. Gardner, M. (1990). *Spectrum: Mathematical magic show.* Washington, D.C., DC: Mathematical Association of America.

PROBLEM

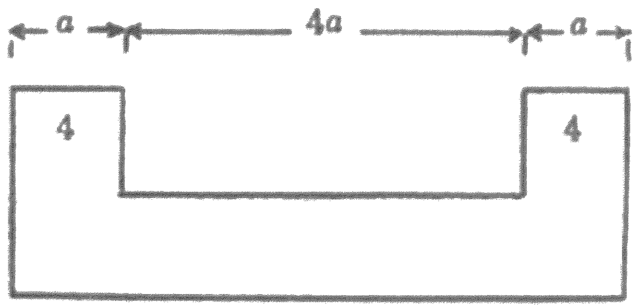
Years ago, my son, then a little boy, was playing with some coloured blocks. There were two of each colour, and one day I noticed that he had placed them in a single pile so that between the red pair there was one block, two between the blue pair, and three between the yellow. I then found that by a complete rearrangement I could add a green pair with four between them.

Clearly this is a perfectly general problem. For convenience we may denote the blocks by a pair of 1's, a pair of 2's etc. By experimenting with pieces of card cut as shown in the diagram, I have obtained the following solutions for n pairs; the other cases with $n \leqslant 15$ I do not believe to be soluble. Can anyone produce a theoretical treatment?

$n = 3$: 312132
$n = 4$: 41312432
$n = 7$: 17126425374635
$n = 8$: 3181375264285746
$n = 11$: $121e257t8395637e48t694$
$n = 12$: $Tt864e975468tT579e|312132$
$n = 15$: $F0e975fTt86579e0F68tTf|41312432$

$t = 10$, $e = 11$, $T = 12$, $0 = 13$, $f = 14$, $F = 15$.

It will be noticed that the last two cases contain the arrangements for $n = 3$ and 4 as separate groups, which can be placed at either end.



C. Dudley Langford

Problem. (1958). *The Mathematical Gazette*, *42*(341), 228–228. https://doi.org/10.2307/3610395

ON LANGFORD'S PROBLEM (1)

By C. J. Priday

For numbers $a \geqslant b \geqslant 1$ we shall denote by (a, b) the set of numbers $b, b+1, \ldots, a$. We shall say that a set S of numbers is *perfect* if there exists a sequence containing just one pair of each of the numbers in S, satisfying the condition: *for every number r in the set, the two r's are separated by exactly r places*, and having no gaps (a *perfect sequence*).

Example 1. (4, 1) is perfect: 41312432.

We shall say that S is *hooked* if there exists a sequence containing the same numbers and satisfying the same condition, but having a gap one place from one end (a *hooked sequence*).

Example 2. (2, 1) is hooked: 121*2.

Example 3. (8, 2) is hooked: 8642752468357*3.

We note that (by juxtaposition of the corresponding sequences) if two sets S_1 and S_2 without common elements are both perfect then so is their union S, and if one is perfect and the other hooked then S is hooked; while if both are hooked then the corresponding sequences can be "hooked together" and so S is perfect.

Example 4. (8, 2) and (1, 1) are hooked, so (8,1) is perfect:

8642752468357131.

Langford's problem (*Math. Gaz.* (1958), p. 228) may be formulated

* The writing of this paper is part of the work made possible by a grant from the Carnegie Corporation of New York for the development of the author's approach to mathematics.

C. J. Priday, & Roy O. Davies. (1959). On Langford's Problem. *The Mathematical Gazette*, *43*(346), 250–255. https://doi.org/10.2307/3610650

as follows: *for what natural numbers a is (a, 1) perfect?* We shall prove the

THEOREM. *For every natural number a, (a, 1) is either perfect or hooked.*

We shall say that S is *looped* if there exist two sequences, each containing the same numbers and satisfying the same condition as before; one having just one gap, two places from one end, and the other having two gaps, one place and two places from one end.

Example 5. (4, 1) is looped:

131423*24; 2412134**3.

Example 6. (5, 2) is looped:

425324*35; 3425324**5.

Example 7. (8, 1) is looped:

56784151647382*32; 567841516472832**3.

We note that if two sets S_1 and S_2 without common elements are both looped then their union is perfect (a sequence of one kind for S_1 and of the other kind for S_2 can be "looped together").

Our result is based on the

LEMMA. *For every natural number a, $(3a, a)$ is both perfect and looped, $(3a + 2, a)$ is hooked and $(3a + 4, a)$ is looped.*

Proof. We exhibit sequences of the required types (the arrangements simplify when $a = 1$ or 2 but as is easily verified they still exist). The semi-colons are inserted to make the structure more visible.

$$3a, 3a-2, \ldots, a;\ 3a-1, 3a-3, \ldots, a+1;$$
$$a, a+2, \ldots, 3a;\ a+1, a+3, \ldots, 3a-1.$$

$$2a+2, 2a+3, \ldots, 3a;\ a, a+1, \ldots, 2a-1;\ 2a+1, a, 2a;$$
$$a+1, 2a+2, a+2, \ldots, 2a-1, 3a;\ *, 2a+1, 2a.$$

$$2a+2, 2a+3, \ldots, 3a;\ a, a+1, \ldots, 2a;$$
$$a, 2a+1, a+1, \ldots, 3a, 2a;\ *, *, 2a+1.$$

$$3a+2, 3a, \ldots, a;\ 3a+1, 3a-1, \ldots, a+3;$$
$$a, a+2, \ldots, 3a+2;\ a+1, a+3, \ldots, 3a+1;\ *, a+1.$$

$$3a+4, 3a+2, \ldots, a;\ 3a+3, 3a+1, \ldots, a+5;$$
$$a, a+2, \ldots, 3a+4;\ a+3, a+1;$$
$$a+5, a+7, \ldots, 3a+3;\ *, a+1, a+3.$$

$$3a+4, 3a+2, \ldots, a; 3a+3, 3a+1, \ldots, a+5; a, a+2, \ldots, 3a+4;$$

Proof of the Theorem. Since every integer is congruent to 0, 2 or 4 modulo 3, the set $(a, 1)$ can be decomposed into sets to which the Lemma applies, possibly together with (1, 1), (2, 1) (both hooked) or (4, 1) (perfect and looped).

Example

$$(341, 1) = (341, 113) \text{ hooked} + (112, 36) \text{ looped} + (35, 11) \text{ hooked} + (10, 2) \text{ looped} + (1, 1) \text{ hooked.}$$

The set (341, 1) is hooked, because the two looped sets may be combined to form a perfect set and two of the hooked sets may be combined to form another, and the resulting two perfect sequences may be juxtaposed with the remaining hooked sequence to form a hooked sequence for (341, 1).

In general, let $(a, 1)$ be decomposed into h hooked sets, l looped sets and p sets both perfect and looped. If l is even, then the looped sets may be combined in pairs to give perfect sets, and the same may be done with all or all but one of the hooked sets (depending on whether h is even or odd), and so $(a, 1)$ is perfect or hooked. If l is odd but $p \neq 0$, then by using one of the perfect-and-looped sets as a looped set we get the same result. This leaves the case where l is odd and $p = 0$.

The decomposition of $(a, 1)$ must then end in one of the five ways considered below, and we show in each case that (possibly after the decomposition has been modified) one of the looped sets can be eliminated, leaving an even number.

(i) (8, 2) hooked + (1, 1) hooked.

Replace by (8, 1), which is looped (example 7), and combine this with a looped set to give a perfect one.

(ii) (10, 2) looped + (1, 1) hooked.

But (10, 2) is also perfect (we write 0 for 10):

647890462572839503.

(iii) (2, 1) hooked.

Replace by (2, 2) + (1, 1) hooked; the sequence 2**2 may be combined with a looped one to give a perfect sequence.

(iv) (5, 1) hooked.

Replace by (5, 2) + (1, 1) hooked. Since (5, 2) is looped (example 6), it may be combined with a looped set to give a perfect one.

(v) (7, 1) looped.

But (7, 1) is also perfect: 17126425374635.

The proof is now complete. We have not proved two things which appear to be true: (i) no set can be both perfect and hooked; (ii) the condition for $(a, 1)$ to be perfect is that a is of the form $4m - 1$ or $4m$.†

C. J. P

ON LANGFORD'S PROBLEM (II)

By Roy O. Davies

The problem is to arrange the numbers 1, 1, 2, 2, ..., n, n in a sequence (without gaps) in such a way that for $r = 1, 2, \ldots, n$ the two r's are separated by exactly r places; for example

41312432.

Priday has shown in the preceding paper that for every n there exists either such a *perfect sequence* (as he calls it) or else a *hooked sequence*, with a gap one place from one end; for example

345131425*2.

Here we shall show that, as Priday conjectured, the perfect sequence exists only if n is of the form $4m - 1$ or $4m$, and the hooked solution otherwise. The method is Bang's, as used by Skolem‡ in solving a problem equivalent to Langford's with a pair of zeros added. Skolem also gave for his problem explicit perfect sequences for the two favourable cases $n = 4m - 1$, $4m$, and we shall exhibit similar but more complicated sequences for both the perfect and hooked cases in Langford's problem. We thus have an alternative proof of Priday's interesting result.

Theorem 1. *If the numbers* 1, 1, 2, 2, ..., n, n *can be arranged in a perfect sequence then* n *is of the form* $4m - 1$ *or* $4m$, *where* m *is an integer, while if they can be arranged in a hooked sequence then* n *is of the form* $4m - 3$ *or* $4m - 2$.

Proof. Perfect Sequence. Let the first r in the sequence be in the a_rth position; then the other is in the position $a_r + r + 1$, and the numbers a_r, $a_r + r + 1$ $(r = 1, 2, \ldots, n)$ are the numbers $1, 2, \ldots, 2n$ in some order. Therefore

$$\sum_{r=1}^{n} (2a_r + r + 1) = \sum_{i=1}^{2n} i = n(2n + 1),$$

whence one deduces that $(3n^2 - n)/4$ equals $\sum a_r$ and is thus an integer. It follows that n is of the form $4m - 1$ or $4m$.

Hooked Sequence. With the same notation as before, the numbers a_r, $a_r + r + 1$ $(r = 1, \ldots, n)$ may be taken to be the numbers $1, 2, \ldots, 2n - 1, 2n + 1$ in some order. Therefore

$$\sum_{r=1}^{n} (2a_r + r + 1) = n(2n + 1) + 1,$$

whence $(3n^2 - n + 2)/4$ equals $\sum a_r$ and is thus an integer. It follows that n is of the form $4m - 3$ or $4m - 2$.

REMARK. Similar arguments show more generally that if a pair of each of some n distinct numbers can be arranged in an perfect sequence (or either kind of "looped sequence" as considered by Priday) then n is of the form $4m - 1$ or $4m$, while if they can be arranged in a hooked sequence then n is of the form $4m - 3$ or $4m - 2$.

THEOREM 2. *The numbers* $1, 1, 2, 2, \ldots, n, n$ *can be arranged in a perfect sequence if* n *is of the form* $4m - 1$ *or* $4m$, *and in a hooked sequence otherwise.*

Proof. We exhibit the required sequences in the four cases. Each consists mostly of strings of consecutive odd or consecutive even numbers and all but the two extreme terms of each such string are replaced by dots below. Suitably interpreted, the sequences are valid, although they degenerate, for $m = 1, 2$.

The case $n = 4m$.

$4m - 4, \ldots, 2m, 4m - 2, 2m - 3, \ldots, 1, 4m - 1, 1, \ldots, 2m - 3, 2m, \ldots, 4m - 4, 4m, 4m - 3, \ldots, 2m + 1, 4m - 2, 2m - 2, \ldots, 2, 2m - 1, 4m - 1, 2, \ldots, 2m - 2, 2m + 1, \ldots, 4m - 3, 2m - 1, 4m.$

The case $n = 4m - 1$.

$4m - 4, \ldots, 2m, 4m - 2, 2m - 3, \ldots, 1, 4m - 1, 1, \ldots, 2m - 3, 2m, \ldots, 4m - 4, 2m - 1, 4m - 3, \ldots, 2m + 1, 4m - 2, 2m - 2, \ldots, 2, 2m - 1, 4m - 1, 2, \ldots, 2m - 2, 2m + 1, \ldots, 4m - 3.$

The case $n = 4m - 2$.

$1, 2m - 3, 1, 4m - 8, \ldots, 2m - 2, 2m - 5, \ldots, 3, 4m - 3, 2m - 3, 4m - 6, 3, \ldots, 2m - 5, 4m - 4, 2m - 2, \ldots, 4m - 8,$

The case $n = 4m - 3$.

$4m-6, \ldots, 2m-2, 4m-5, 2m-5, \ldots, 1, 4m-4, 1, \ldots, 2m-5, 2m-2, \ldots, 4m-6, 4m-3, 4m-7, \ldots, 2m-1, 4m-5, 2m-4, \ldots, 2, 2m-3, 4m-4, 2, \ldots, 2m-4, 2m-1, \ldots, 4m-7, 2m-3, *, 4m-3.$

REMARKS. By adding a pair of adjacent zeros at one end, we obtain an alternative solution to the previously mentioned problem of Skolem, and a solution to a problem stated by him (but not solved) in a more recent paper.†

It would be interesting to know roughly how many different solutions of Langford's problem exist for large n. They are surprisingly numerous even for $n = 7$: namely, 25 distinct perfect sequences, not counting as distinct a sequence and the same one in reverse order. For $n = 3$ and $n = 4$ there is only one solution.

The University, Leicester R. O. D.

Editorial Note: Solutions to parts or the whole of Langford's problem have also been submitted by F. Downton, R. Sibson, R. A. Bull and J. R. A. Copper.

GLEANINGS FAR AND NEAR

1933. The post of Lord Great Chamberlain has existed for some 850 years; his is the only hereditary office that can go through the female line. Because of one or two disputed claims, three families share the office in rotation; or, rather, the Cholmondeleys have every alternate reign, the Ancasters and Caringtons every third.—*The Observer.* 3 November 1957. [Per Mr. R. F. Wheeler.]

1934. The first Russian satellite ... weighed 184 lb, nearly ten orders of magnitude heavier than the scheduled American vehicle.—*Discovery.* November 1957. [Per Mr. R. F. Wheeler.]

1935. Do you know how many ways there are to play the first four moves in a game of chess? Each player has sixteen units at his disposal. Offhand you might say there are 100 or 200 different ways to play these units in the first four moves. Yet the mathematicians tell us that the number of possible ways is no less than 318,979,654,000! But that's nothing. By the time you get to the problem of how many different ways there are to play the first *ten* moves, the number has risen to the staggering figure of 169,518,829,100,544,000,000,000,000,000!! Even if the experts have dropped a logarithm or two, and are out by a few billion possibilities or so, they have succeeded in making their point: chess can be a mighty complicated game. Yet for at least 800 years there have been experts who could play chess blindfold!—From an article on blindfold chess in the American magazine "*Chess Review*" 1951. [Per Mr. W. H. Cozens.]

Mathematical Games

This page gives the exact text of Martin Gardner's columns relating to Langford's Problem. Note that he revisits the problem in *Mathematical Magic Show*, published by Alfred A. Knopf, ISBN 0-88385-449-X, first and second editions.

November 1967, pages 127-128

6. Many years ago C. Dudley Langford, a Scottish Mathematician, was watching his little boy play with colored blocks. There were two blocks of each color, and the child had piled six of them in a column in such a way that one block was between the red pair, two blocks were between the blue pair and three were between the yellow pair. Substitute digits 1, 2, 3 for the colors and the sequence can be represented as 312132.

This is the unique answer (not counting its reversal as being different) to the problem of arranging the six digits so that there is one digit between the 1's and there are two digits between the two 2's and three digits between the two 3's.

Langford tried the same task with four pairs of differently colored blocks and found that it too had a unique solution. Can the reader discover it? A convenient way to work on this easy problem is with eight playing cards: two aces, two deuces, two threes and two fours. The object is to place them in a row so that one card separates the aces, two cards separate the deuces, and so on.

There are no solutions to "Langford's problem", as it is now called, with five or six pairs of cards. There are 25 distinct solutions with seven pairs. No one knows how to determine the number of distinct solutions for a given number of pairs except by exhaustive trial-and-error methods, but perhaps the reader can discover a simple method of determining if there *is* a solution. Next month I shall make some remarks about the general case and cite major references.

December 1967, pages 131-132

The unique solution to Langford's Problem with four pairs of cards is 41312432. It can be reversed, of course, but this is not considered a different solution. If *n* is the number of pairs, the problem has a solution only if *n* is a multiple of four or one less than such a multiple.

Gardner cites references for Langford, Priday, Davies, Gillespie and Utz, and Nickerson as given in the bibliography.

11.

Combinatorische Aufgabe.

(Von Herrn Professor Dr. *J. Steiner* zu Berlin.)

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a) Welche Zahl, *N*, von Elementen hat die Eigenschaft, dafs sich die Elemente so zu dreien ordnen lassen, dafs je zwei in *einer*, aber *nur in einer* Verbindung vorkommen? Wie viele wesentlich verschiedene Anordnungen, d. h. solche, die nicht durch eine blofse Permutation der Elemente auseinander hervorgehen, giebt es bei jeder Zahl?

b) Wenn ferner die Elemente sich so zu vieren verbinden lassen sollen, dafs jede drei freien Elemente, d. h. solche, welche nicht schon einen der vorigen Dreier (*a*.) bilden, immer in *einem* aber *nur in einem* Vierer vorkommen, und dafs auch keine 3 Elemente eines solchen Vierers einem der vorigen Dreier angehören; so entsteht daraus keine neue Bedingung für die Zahl *N*.

c) Sollen die Elemente sich weiter so zu Fünfern combiniren lassen, dafs je vier unter sich noch freie Elemente, d. h. welche keinen der zuvor gebildeten Vierer (*b*.) ausmachen, noch einen der früheren Dreier (*a*.) enthalten, immer in einem, aber nur in einem Fünfer vorkommen, und dafs ein solcher Fünfer keinen der schon gebildeten Dreier noch Vierer enthält: welche neue Modification erleidet dann die Zahl *N*?

d) Und sollen die Elemente sich ähnlicherweise so zu Sechsern verbinden lassen, dafs zu je fünf unter sich noch freien Elementen ein bestimmtes sechstes gehört, aber keiner der so gebildeten Sechser einen der früheren Dreier oder Vierer oder Fünfer enthält; welche Beschränkung erleidet dann die Zahl *N*?

e) Eben so sollen Siebner gebildet werden, so dafs zu je sechs unter sich freien Elementen ein bestimmtes siebentes gehört, aber ein solcher Siebner weder einen der vorigen Dreier, noch Vierer, noch Fünfer, noch Sechser enthält. Und so soll fortgefahren werden, bis etwa für die Zahl *N* die Unmöglichkeit höherer Verbindungen dieser Art eintritt. Zudem soll auf jeder Stufe die allgemeine Form der Zahl *N*, für welche die geforderten Combinationen möglich sind, angegeben, so wie umgekehrt gezeigt werden, ob bei jeder Zahl von der

Combinatorische Aufgabe, Steiner, J., *J. reine angew. Math.*, 1853, v45, 181-2, (Gesammelte Werke II, Berlin, 1882, 437-438)

Mathematical Magic Show, Page 76

6. LANGFORD'S PROBLEM

MANY YEARS AGO C. Dudley Langford, a Scottish mathematician, was watching his little boy play with colored blocks. There were two blocks of each color, and the child had piled six of them in a column in such a way that one block was between the red pair, two blocks were between the blue pair, and three were between the yellow pair. Substitute digits 1, 2, 3 for the colors and the sequence can be represented as 312132.

This is the unique answer (not counting its reversal as being different) to the problem of arranging the six digits so that there is one digit between the 1's and there are two digits between the e's and three digits between the 3's.

Langford tried the same task with four pairs of differently colored blocks and found that it too had a unique solution. Can the reader discover it? A convenient way to work on this easy problem is with eight playing cards: two aces, two deuces, two threes, and two fours. The object then is to place them in a row so that one card separates the aces, two cards separate the deuces, and so on.

There are no solutions to "Langford's problem", as it is now called, with five or six pairs of cards. There are 26 distinct solutions with seven pairs. No one knows how to determine the number of distinct solutions for a given number of pairs except by exhaustive trial-and-error methods, but perhaps the reader can discover a simple method of determining if there is a solution.

Mathematical Magic Show, pages 77-78, (some answers)

6. The unique solution to Langford's problem with four pairs of cards is 41312432. It can be reversed, of course, but this is not considered a different solution.

If n is the number of pairs, the problem has a solution only if n is a multiple of 4 or one less than such a multiple. C. Dudley Langford posed his problem in *The Mathematical Gazette* (Vol. 42, October 1958, page 228). For subsequent discussion see C. J. Priday, "On Langford's Problem (I)", and Roy 0. Davies, "On Langford's Problem (II)", both in *The Mathematical Gazette* (Vol.43, December 1959, pages 250-55).

The 26 solutions for n = 7 are given in *The Mathematical Gazette* (Vol. 55, February 1971, page 73). Numerous computer programs have confirmed this list, and found 150 solutions for n = 8. E. J. Groth and John Miller independently ran programs which agreed on 17,792 sequences for n = 11, and 108,144 for n = 12.

R. S. Nickerson, in "A Variant of Langford's Problem", *American Mathematical Monthly* (Vol. 74, May 1967, pages 591-95), altered the rules so that the second card of a pair, each with value k, is the kth card after the first card; put another way, each pair of value k is separated by k-1 cards. Nickerson proved that the problem was solvable if and only if the number of pairs is equal to 0 or 1 (modulo 4). John Miller ran a program which found three solutions for n=4 (they are 11423243, 11342324, and 41134232); five solutions for n=5; 252 solutions for n=8; and 1,328 for n=9.

Frank S. Gillespie and W. R. Utz, in "A Generalized Langford Problem", *Fibonacci Quarterly* (Vol. 4, April 1966, pages 184-86), extended the problem to triplets, quartets, and higher sets of cards. They were unable to find solutions for any sets higher than pairs. Eugene Levine, writing in the same journal ("On the Generalized Langford Problem", Vol. 6, November 1968, pages 135-38), showed that a necessary condition for a solution in the case of triplets is that n (the number of triplets) be equal to -1, 0, or 1 (modulo 9). Because he found solutions for n = 9, 10, 17, 18, and 19, he conjectured that the condition is also sufficient when n exceeds 8. The nonexistence of a solution for n=8 was later confirmed by a computer search.

Levine found only one solution for n=9. I know of no other solution; perhaps it is unique. Readers may enjoy finding it. Take from a deck all the cards of three suits which have digit values (ace through nine). Can you arrange these 27 cards in a row so that for each triplet of value k cards there are k cards between the first and second card, and k cards between the second and third? It is an extremely difficult combinatorial puzzle.

D. P. Roselle and T. C. Thomasson, Jr., "On Generalized Langford Sequences", *Journal of Combinatorial Theory* (Vol. 11, September 1971, pages 196-99), report on some new non-existence theorems, and give one solution each for triplets when n = 9, 10, and 17. So far as I am aware, no Langford sequence has yet been found for sets of integers higher than three, nor has anyone proved that such sequences do or do not exist.

Gardner, M. (1990). Spectrum: Mathematical magic show. Washington, D.C., DC: Mathematical Association of America.